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Two-scale Finite Element Discretizations for Semilinear Parabolic Equations

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Abstract: In this paper, to reduce the computational cost of solving semilinear parabolic equations on a tensor product domain $\Omega \subset \mathbb{R}^d$ with $d = 2$ or 3, some two-scale finite element discretizations are proposed and analyzed. The time derivative in semilinear parabolic equations is approximated by the backward Euler finite difference scheme. The two-scale finite element method is designed for the space discretization. The idea of the two-scale finite element method is based on an understanding of a finite element solution to an elliptic problem on a tensor product domain. The high frequency parts of the finite element solution can be well captured on some univariate fine grids and the low frequency parts can be approximated on a coarse grid. Thus the two-scale finite element approximation is defined as a linear combination of some standard finite element approximations on some univariate fine grids and a coarse grid satisfying $H = O(h^{1/2})$, where h and H are the fine and coarse mesh widths, respectively. It is shown theoretically and numerically that the backward Euler two-scale finite element solution not only achieves the same order of accuracy in the $H^1(\Omega)$ norm as the backward Euler standard finite element solution, but also reduces the number of degrees of freedom from $O(h^{-d} \times \tau^{-1})$ to $O(h^{-(d+1)/2} \times \tau^{-1})$ where τ is the time step. Consequently the backward Euler two-scale finite element method for semilinear parabolic equations is more efficient than the backward Euler standard finite element method.

Keywords: Two-scale, Finite Element, Combination, Semilinear Parabolic Equation

1. Introduction

Consider the following semilinear parabolic equation:

$$\begin{cases} u_t - \Delta u = f(u), & (X,t) \in \Omega \times J, \\ u = 0, & (X,t) \in \partial\Omega \times J, \\ u(0) = u_0, & (X,t) \in \Omega \times \{t = 0\}, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a tensor-product domain $(d = 2, 3)$, $J = (0, T]$ and $u_t = \frac{\partial u}{\partial t}$. $f(.)$ is twice continuously differentiable and u_0 is a given smooth function.

The mathematical theory of Galerkin finite element methods for parabolic equation has been discussed systematically in [37]. Superclose properties of finite element method for linear parabolic problem were studied in [38]. The hp-version discontinuous Galerkin finite element method was proposed for semilinear parabolic problems in [20]. To reduce computational cost, the two-grid methods with finite difference and finite volume methods were proposed for nonlinear parabolic problems [6, 7, 11, 28]. Two-grid finite element methods for nonlinear Schrödinger equations were developed in [19, 46]. Two-grid mixed finite element methods for nonlinear parabolic problems were studied in [8, 10]. Superconvergence properties of two-grid finite element methods for semilinear parabolic problems were analyzed in [35, 36]. The error estimates of the two-grid discontinuous Galerkin method for nonlinear parabolic equations were studied in [42].

To reduce computational complexity and storage requirements further, in this paper we propose some two-scale finite element discretizations for the semilinear parabolic problem (1). That is, the time derivative in (1) is approximated by the backward Euler finite difference scheme. The two-scale finite element method is used for the space discretization. The two-scale finite element approximations are constructed by using a linear combination of some standard finite element approximations on a coarse grid and some univariate fine grids. The main idea of the two-scale discretizations is based on an understanding of the frequency resolution of a finite element solution to an elliptic problem. For a solution to an elliptic problem, high frequency components can be well approximated on a fine grid and low frequency components can be computed on a relatively coarse grid (see, e.g., [1, 39, 40, 41, 44]). Moreover, it is known that for elliptic problems on tensor product domains, some high frequencies involve a tensor product of univariate low frequencies, hence they can be handled numerically by a tensor product of univariate fine and coarse grids [4, 5, 15, 25, 30, 31, 34]. It will be shown that on choosing $H = O(h^{1/2})$ the backward Euler two-scale finite element method achieves the same order of accuracy as the backward Euler standard finite element method while reducing the number of degrees of freedom from $O(h^{-d} \times \tau^{-1})$ to $O(h^{-(d+1)/2} \times \tau^{-1})$ where τ is the time step for time discretization of (1).

The two-scale finite element method is related to the multi-level sparse grid method [45]. To reduce computational cost, the sparse grid method was proposed for elliptic boundary value problems in which the multi-level basis was used [4, 5, 14, 16, 17, 25, 33, 34, 43]. The combination technique [12, 15], which can be viewed as a variant of the sparse grid method, has been developed. The two-scale finite element discretization uses the two-level basis instead of the multi-level basis [1, 43]. It has been proposed for linear boundary value and eigenvalue problems [13, 18, 30, 31]. The two-level basis is more flexible than the multi-level basis [25, 30].

The so-called superconvergence technique [13, 25, 30, 31, 34, 47] is used in analysis for the two-scale finite element approximations. It has been applied to obtain asymptotic error expansions of the finite element solutions by Lin et al. [2, 23, 26, 27]. A related method is the so-called splitting extrapolation method [21, 22, 48], which is based on the multi-parameter asymptotic error expansions.

This paper is structured as follows. In Section 2, some basic notation is presented. In Section 3, we present some tensor product operators. A standard Galerkin finite element method with the backward Euler finite difference scheme is described. The related error estimate is proposed. In Section 4, the two-scale finite element method with the backward Euler finite difference scheme for semilinear parabolic problem is presented. The theoretical analysis shows that the two-scale finite element method is more efficient than the standard finite element method. Numerical results to illustrate our theory are presented in Section 5. Finally some concluding remarks are given in Section 6.

2. Preliminaries

Let $\Omega = (0,1)^d$ $(d = 2,3)$ and ϖ be any measurable subset of Ω[1]. We will use standard notations for the Sobolev spaces $W^{s,p}(\varpi)$ and their associated norms and seminorms [9]. When $p = 2$, let $H^s(\varpi) = W^{s,2}(\varpi)$ and $\|\cdot\|_{s,\varpi} = \|\cdot\|_{s,2,\varpi}$. Let $H_0^1(\Omega) = \{v \in H^1(\Omega) : v\mid_{\partial\Omega} = 0\}$, where $v\mid_{\partial\Omega} = 0$ in the sense of traces. Let $H^{-1}(\Omega)$ be the dual space of $H_0^1(\Omega)$. Write $(\cdot,\cdot)$ for the $L^2(\Omega)$ inner product.

Write $\mathbb{N}_0$ for the set of all nonnegative integers. Let $\mathbb{Z}_d = \{1, 2, \dots, d\}$. For each function $w \in W^{s,p}(\Omega)$, set

$$(D^\alpha w)(x) = \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d} w}{\partial x_d^{\alpha_d}}\right)(x),$$

where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}_0^d$ and $x = (x_1, x_2, \dots, x_d) \in \Omega$. For general d-dimensional vector $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, let $\mathbf{x}^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$ and $\mathbf{x}\alpha = (x_1\alpha_1, \dots, x_d\alpha_d)$.

For a real number $t \in \mathbb{R}$, let

$$t\alpha = (t\alpha_1, \dots, t\alpha_d).$$

The notation $\alpha \le \beta$ denotes $\alpha_i \le \beta_i$ for $i \in \mathbb{Z}_d$, $\alpha, \beta \in \mathbb{N}_0^d$ throughout this paper.

We need the so-called mixed Sobolev space:

$$W^{G,3}(\omega) := \{v \in H^2(\omega) :\ D^\alpha v \in L^2(\omega),\ \mathbf{0} \le \alpha \le 2\mathbf{e},\ |\alpha| = 3\},$$

with its natural norms $\|\cdot\|_{W^{G,3}(\omega)}$ [34], where $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$, $\mathbf{e} = (1, \dots, 1) \in \mathbb{R}^d$ and $|\alpha| = \alpha_1 + \dots + \alpha_d$. Let $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^d$ for each $i \in \mathbb{Z}_d$, which means only the i^{th} component of $\mathbf{e}_i$ is 1 and all other ones are 0. Set $\hat{\mathbf{e}}_i = \mathbf{e} - \mathbf{e}_i$. The notation $A \lesssim B$ means $A \le CB$ for constant C which only depends on the data of the problem and does not depend on mesh parameters. We use the notation $A = \mathcal{O}(B)$ to denote $|A| \lesssim B$.

3. A Galerkin Finite Element Method

The variational form of (1) is to find $u : J \mapsto H_0^1(\Omega)$ such that

$$\begin{cases} (u_t, v) + (\nabla u, \nabla v) = (f(u), v), & \forall v \in H_0^1(\Omega), \\ u(X, 0) = u_0(X), & X \in \Omega, \end{cases} \tag{2}$$

where $(f(u), v) = \int_\Omega fv$.

Let $T^h[0,1]$ be the uniform mesh, where $h = 1/N \in [0,1]$ is the mesh size with $N \in \mathbb{N}_0/\{0\}$. Let $S^h[0,1] \subset C[0,1]$ be the associated piecewise linear finite element space.

Set $\mathbf{h} = (h_1, \dots, h_d)$ and $h = \max_{1\le j\le d} h_j$ with $h_j = 1/N_j$ for $N_j \in \mathbb{N}_0/\{0\}$, $j \in \mathbb{Z}_d$. Define a tensor-product mesh on $\bar{\Omega} = [0,1]^d$ by

$$T^{\mathbf{h}}(\Omega) := T^{h_1}[0,1] \times \dots \times T^{h_d}[0,1].$$

[1] Though our analysis is for Ω, it can be generalized to any tensor product domain.

The corresponding tensor product spaces of piecewise d-linear functions on Ω are

$$S^{\mathbf{h}}(\Omega) := S^{h_1}[0,1] \otimes \cdots \otimes S^{h_d}[0,1]$$

and

$$S_0^{\mathbf{h}}(\Omega) := S^{\mathbf{h}}(\Omega) \cap H_0^1(\Omega).$$

The standard Lagrange interpolation operator $I_{\mathbf{h}} : C(\bar{\Omega}) \to S^{\mathbf{h}}(\Omega)$ is defined by

$$I_{\mathbf{h}} := I_{h_1\mathbf{e}_1} \circ \cdots \circ I_{h_d\mathbf{e}_d} \equiv \prod_{i=1}^{d} I_{h_i\mathbf{e}_i}.$$

Here, $I : C(\bar{\Omega}) \to C(\bar{\Omega})$ denotes the identity operator.
The Ritz projector $R_{\mathbf{h}} : H_0^1(\Omega) \mapsto S_0^{\mathbf{h}}(\Omega)$ is defined by

$$(\nabla(w - R_{\mathbf{h}}w), \nabla v) = 0, \quad \forall v \in S_0^{\mathbf{h}}(\Omega), \tag{3}$$

for each $w \in H_0^1(\Omega)$. There holds that [3, 9, 23, 25, 48]

$$\|w - R_{\mathbf{h}}w\|_{0,\Omega} + h\|w - R_{\mathbf{h}}w\|_{1,\Omega} \lesssim h^2\|w\|_{2,\Omega}, \tag{4}$$

$$\|I_{\mathbf{h}}w - R_{\mathbf{h}}w\|_{1,\Omega} \lesssim h^2\|w\|_{W^{G,3}(\Omega)}, \tag{5}$$

for each $w \in H_0^1(\Omega) \bigcap W^{G,3}(\Omega)$.

Let $\{t_n | t_n = n\tau,\ 0 \le n \le N\}$ be a uniform partition in time with time step τ and $u^n = u(X, t_n)$. For a sequence of a functions $\{\phi^n\}_{n=0}^N$, we denote $D_\tau\phi^n = (\phi^n - \phi^{n-1})/\tau$. Then with the mesh and trial space in Section 2, the backward Euler scheme of (2) is: find $u_{\mathbf{h}}^n \in S_0^{\mathbf{h}}(\Omega)$ for $n = 1, 2, \cdots, N$ such that

$$\begin{cases} (D_\tau u_{\mathbf{h}}^n, v) + (\nabla u_{\mathbf{h}}^n, \nabla v) = (f(u_{\mathbf{h}}^n), v_{\mathbf{h}}) & \forall v \in S_0^{\mathbf{h}}(\Omega), \\ u_{\mathbf{h}}^0 = R_{\mathbf{h}}u_0(X), \quad X \in \Omega. \end{cases} \tag{6}$$

The following Lemma for superclose property is a generalization of Theorem 2.1 in [36] where only two-dimensional case is mentioned and $H^3(\Omega)$ instead of $W^{G,3}(\Omega)$ is used.

Lemma 3.1. Let u and $u_{\mathbf{h}}^n$ be the solution of (2) and (6), respectively. Assume that $u \in L^\infty(J; W^{G,3}(\Omega))$, $u_t \in L^2(J; H^2(\Omega))$, and $u_{tt} \in L^\infty(J; L^2(\Omega))$, then

$$\|u_{\mathbf{h}}^n - I_{\mathbf{h}}u^n\|_{1,\Omega} \le Ch^2(\|u\|^2_{L^\infty(J;W^{G,3}(\Omega))} + \|u_t\|_{L^2(J;H^2(\Omega))})^{1/2} + C\tau\|u_{tt}\|_{L^\infty(J;L^2(\Omega))}. \tag{7}$$

Proof. By (5) and the proof of Theorem 2.1 in [36], we can obtain this conclusion directly.

By Lemma 3.1 we have the following result immediately. Similar result has been proposed in [35] for two-dimensional case with different proof.

Theorem 3.1. Let u and $u_{\mathbf{h}}^n$ be the solution of (2) and (6), respectively. Assume that $u \in L^\infty(J; W^{G,3}(\Omega))$, $u_t \in L^2(J; H^2(\Omega))$, and $u_{tt} \in L^\infty(J; L^2(\Omega))$, then

$$\|u^n - u_{\mathbf{h}}^n\|_{1,\Omega} \lesssim h + \tau. \tag{8}$$

4. Two-scale Finite Element Method

In this section, we propose two-scale finite element discretizations for the semilinear parabolic problem.

Let $h, H \in (0,1)$ and assume that H/h is a positive integer. In practice we choose $h \ll H$. Let $w_{h\alpha+H(\mathbf{e}-\alpha)} \in S_0^{h\alpha+H(\mathbf{e}-\alpha)}(\Omega)$ for $\mathbf{0} \le \alpha \le \mathbf{e}$. Following [29, 30, 32], we define a Boolean sum as follows

$$B_{H\mathbf{e}}^h w_{h\mathbf{e}} = \sum_{i=1}^{d} w_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} - (d-1)w_{H\mathbf{e}}, \quad \forall w_{h\mathbf{e}} \in S_0^{h\mathbf{e}}(\Omega), \tag{9}$$

in which each summand uses a mesh of width h in at most one coordinate direction. For example, if $d = 2$, we have $B_{H,H}^h w_{h,h} = w_{h,H} + w_{H,h} - w_{H,H}$. If $d = 3$, we have $B_{H,H,H}^h w_{h,h,h} = w_{h,H,H} + w_{H,h,H} + w_{H,H,h} - 2w_{H,H,H}$.

In our analysis, we need the following result [13, 22, 24, 29, 30].

Proposition 4.1. If $w \in C(\bar{\Omega}) \cap W^{G,3}(\Omega)$, then

$$H\|B_{H\mathbf{e}}^h I_{h\mathbf{e}}w - I_{h\mathbf{e}}w\|_{1,\Omega} + \|B_{H\mathbf{e}}^h I_{h\mathbf{e}}w - I_{h\mathbf{e}}w\|_{0,\Omega} \lesssim H^3\|w\|_{W^{G,3}(\Omega)}. \tag{10}$$

Our first algorithm is the basic two-scale finite element discretization with the backward Euler finite difference scheme, in which at each time step the two-scale finite element solution is a linear combination of standard finite element solutions on a coarse grid and some univariant fine grids.

Algorithm 4.1. For $n = 1, 2, \cdots, N$,

1. find $u_{H\mathbf{e}}^n \in S_0^{H\mathbf{e}}(\Omega)$ such that

$$\begin{cases} (D_\tau u_{H\mathbf{e}}^n, v) + (\nabla u_{H\mathbf{e}}^n, \nabla v) = (f(u_{H\mathbf{e}}^n), v) & \forall v \in S_0^{H\mathbf{e}}(\Omega), \\ u_{H\mathbf{e}}^0 = R_{H\mathbf{e}}u_0, \end{cases}$$

and for each $i \in \mathbb{Z}_d$, find $u_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}^n \in S_0^{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}(\Omega)$ such that

$$\begin{cases} (D_\tau u_{h\mathbf{e}_i+H\hat{\mathbf{e}}_i}^n, v) + (\nabla u_{h\mathbf{e}_i+H\hat{\mathbf{e}}_i}^n, \nabla v) = (f(u_{h\mathbf{e}_i+H\hat{\mathbf{e}}_i}^n), v) & \forall v \in S_0^{h\mathbf{e}_i+H\hat{\mathbf{e}}_i}(\Omega), \\ u_{h\mathbf{e}_i+H\hat{\mathbf{e}}_i}^0 = R_{h\mathbf{e}_i+H\hat{\mathbf{e}}_i}u_0. \end{cases}$$

2. (Two-scale solution) Set

$$(\check{u}^h_{H\mathbf{e}})^n = B^h_{H\mathbf{e}} u^n_{h\mathbf{e}} = \sum_{i=1}^d u^n_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} - (d-1) u^n_{H\mathbf{e}}.$$

Theorem 4.1. Assume that $u \in H^1_0(\Omega) \cap W^{G,3}(\Omega)$ and u is the solution of (2), then

$$\|u^n - (\check{u}^h_{H\mathbf{e}})^n\|_{1,\Omega} \lesssim h + H^2 + \tau. \tag{11}$$

Proof. Because $C(\bar{\Omega}) \cap H^1_0(\Omega) \cap W^{G,3}(\Omega)$ is dense in $H^1_0(\Omega) \cap W^{G,3}(\Omega)$, we only need to prove (11) for $u \in C(\bar{\Omega}) \cap H^1_0(\Omega) \cap W^{G,3}(\Omega)$. Using the definition $(\check{u}^h_{H\mathbf{e}})^n = B^h_{H\mathbf{e}} u^n_{h\mathbf{e}}$, we have

$$\begin{aligned}\|u^n_{h\mathbf{e}} - (\check{u}^h_{H\mathbf{e}})^n\|_{1,\Omega} &\le \|I_{h\mathbf{e}} u^n - u^n_{h\mathbf{e}}\|_{1,\Omega} + \sum_{i=1}^d \|I_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} u^n - u^n_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}\|_{1,\Omega} \\ &\quad + (d-1)\|I_{H\mathbf{e}} u^n - u^n_{H\mathbf{e}}\|_{1,\Omega} + \|I_{h\mathbf{e}} u^n - B^h_{H\mathbf{e}} I_{h\mathbf{e}} u^n\|_{1,\Omega} \\ &\lesssim H^2 + \tau,\end{aligned}$$

where Lemma 3.1 and Proposition 4.1 are used in the last inequality. Thus, we have

$$\|u^n - (\check{u}^h_{H\mathbf{e}})^n\|_{1,\Omega} \lesssim h + H^2 + \tau.$$

Our second two-scale finite element discretization with the backward Euler finite difference scheme is described in Algorithm 4.2. In this algorithm, at each time step we solve a semilinear system on a coarse grid and some linear systems on some partially fine grids.

Algorithm 4.2. For $n = 1, 2, \cdots, N$,

1. Compute semilinear problem on the coarse grid: find $u^n_{H\mathbf{e}} \in S^{H\mathbf{e}}_0(\Omega)$ such that

$$\begin{cases} (D_\tau u^n_{H\mathbf{e}}, v) + (\nabla u^n_{H\mathbf{e}}, \nabla v) = (f(u^n_{H\mathbf{e}}), v) \quad \forall v \in S^{H\mathbf{e}}_0(\Omega), \\ u^0_{H\mathbf{e}} = R_{H\mathbf{e}} u_0. \end{cases}$$

2. Compute linear problems on some partially fine grids in parallel: for each $i \in \mathbb{Z}_d$, find $\tilde{u}^n_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} \in S^{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}_0(\Omega)$ such that

$$\begin{cases} (D_\tau \tilde{u}^n_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}, v) + (\nabla \tilde{u}^n_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}, \nabla v) = (f(u^n_{H\mathbf{e}}) + f'(u^n_{H\mathbf{e}})(\tilde{u}^n_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} - u^n_{H\mathbf{e}}), v) \quad \forall v \in S^{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}_0(\Omega), \\ \tilde{u}^0_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} = R_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} u_0. \end{cases}$$

3. (Two-scale solution) Set

$$(\tilde{u}^h_{H\mathbf{e}})^n = \sum_{i=1}^d \tilde{u}^n_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} - (d-1) u^n_{H\mathbf{e}}.$$

Theorem 4.2. Assume that $u \in H^1_0(\Omega) \cap W^{G,3}(\Omega)$ and u is the solution of (2), then

$$\|u^n - (\tilde{u}^h_{H\mathbf{e}})^n\|_{1,\Omega} \lesssim h + H^2 + \tau. \tag{12}$$

Proof. Again we only need to prove (12) for $u \in C(\bar{\Omega}) \cap H^1_0(\Omega) \cap W^{G,3}(\Omega)$ since $C(\bar{\Omega}) \cap H^1_0(\Omega) \cap W^{G,3}(\Omega)$ is dense in $H^1_0(\Omega) \cap W^{G,3}(\Omega)$. Using the definition of $(\tilde{u}^h_{H\mathbf{e}})^n$, we have

$$\begin{aligned}\|u^n_{h\mathbf{e}} - (\tilde{u}^h_{H\mathbf{e}})^n\|_{1,\Omega} &= \|\sum_{i=1}^d \tilde{u}^n_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} - (d-1) u^n_{H\mathbf{e}} - u^n_{h\mathbf{e}}\|_{1,\Omega} \\ &\le \sum_{i=1}^d \|\tilde{u}^n_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} - u^n_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}\|_{1,\Omega} + \|u^n_{h\mathbf{e}} - B^h_{H\mathbf{e}} u^n_{h\mathbf{e}}\|_{1,\Omega}.\end{aligned}$$

By Theorem 3.1 in [36], which can be generalized directly for three-dimensional case, we have

$$\|\tilde{u}^n_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} - u^n_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}\|_{1,\Omega} \lesssim \|\tilde{u}^n_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} - I_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} u^n\|_{1,\Omega} + \|I_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i} u^n - u^n_{h\mathbf{e}_i + H\hat{\mathbf{e}}_i}\|_{1,\Omega} \lesssim H^2 + \tau.$$

Thus, we have

$$\|u^n - (\tilde{u}_{H\mathbf{e}}^h)^n\|_{1,\Omega} \lesssim h + H^2 + \tau.$$

Remark 4.1. Comparing Theorem 3.1 with Theorems 4.1 and 4.2, we can see that if we choose $h = O(H^2)$ then the approximate solutions $(\check{u}_{H\mathbf{e}}^h)^n$ and $(\tilde{u}_{H\mathbf{e}}^h)^n$, obtained from Algorithms 4.1 and 4.2, can achieve the same accuracy as $u_{h\mathbf{e}}^n$ from the standard discretization (6), while both $(\check{u}_{H\mathbf{e}}^h)^n$ and $(\tilde{u}_{H\mathbf{e}}^h)^n$ require only $O(h^{-(d+1)/2} \times \tau^{-1})$ degrees of freedom compared with $O(h^{-d} \times \tau^{-1})$ degrees of freedom required by $u_{h\mathbf{e}}^n$. Hence the two-scale finite element discretizations (Algorithms 4.1 and 4.2) are more efficient than the standard finite element discretization (6). Moreover, in Algorithm 4.2 one only need to solve the semilinear parabolic problem on the coarse mesh, hence Algorithm 4.2 is even better than Algorithm 4.1.

5. Numerical Examples

In this section, some numerical examples are presented to show the efficiency of the two-scale finite element method. To optimize the computational cost, in all of the numerical experiments we choose $h = H^2$ and $\tau = h$.

Example 1. Consider the following semilinear parabolic equation:

$$\begin{cases} u_t - \Delta u - \sin u = g, & (X,t) \in \Omega \times J, \\ u = 0, & (X,t) \in \partial\Omega \times J, \\ u(0) = u_0, & (X,t) \in \Omega \times \{t = 0\}, \end{cases} \quad (13)$$

where $\Omega = (0,1) \times (0,1)$ and $J \in (0,1]$. g and u_0 are computed from the exact solution

$$u(x,y,t) = e^{-t} \sin(\pi x) \sin(\pi y).$$

In Tables 1 and 2, the numerical results at $t = 0.5$ and $t = 1.0$ are shown, respectively. The approximate solution $u_{h,h}^n$ is obtained from the standard finite element discretization (6) while $(\check{u}_{H,H}^h)^n$ and $(\tilde{u}_{H,H}^h)^n$ are computed by Algorithms 4.1 and 4.2, respectively. These results support Theorems 3.1, 4.1, and 4.2. It can be seen that the two-scale finite element solutions $(\check{u}_{H,H}^h)^n$ and $(\tilde{u}_{H,H}^h)^n$ achieves the same order of accuracy as the standard finite element solution $u_{h,h}^n$, but with much less computational cost. For example, when $h = 1/100$, at each time step, the exact degrees of freedom to get $(\check{u}_{H,H}^h)^n$ and $(\tilde{u}_{H,H}^h)^n$ is only 100×10 while that for the standard finite element solution is 100×100. Thus the two-scale finite element discretizations are more efficient than the standard finite element discretization. Moreover, in the process of computing $(\tilde{u}_{H,H}^h)^n$, one only need solve semilinear problem on the coarse grid. Hence it is even better than $(\check{u}_{H,H}^h)^n$.

Table 1. *The errors at* $t = 0.5$. *Compare the two-scale finite element method with the standard finite element method.*

$\mathbf{1/h \times 1/H}$	$\|\mathbf{u^n - u^n_{h,h}}\|_1$	$\|\mathbf{u^n - (\check{u}^h_{H,H})^n}\|_1$	$\|\mathbf{u^n - (\tilde{u}^h_{H,H})^n}\|_1$
4×2	0.30343903	0.31174759	0.31173088
16×4	0.07635569	0.07704568	0.07704581
64×8	0.01910041	0.01914644	0.01914647
100×10	0.01222476	0.01224376	0.01224377
convergence rate	$\mathcal{O}(h)$	$\mathcal{O}(h)$	$\mathcal{O}(h)$

Table 2. *The errors at* $t = 1.0$. *Compare the two-scale finite element method with the standard finite element method.*

$\mathbf{1/h \times 1/H}$	$\|\mathbf{u^n - u^n_{h,h}}\|_1$	$\|\mathbf{u^n - (\check{u}^h_{H,H})^n}\|_1$	$\|\mathbf{u^n - (\tilde{u}^h_{H,H})^n}\|_1$
4×2	0.18404477	0.18907448	0.18907027
16×4	0.04631234	0.04673028	0.04673031
64×8	0.01158504	0.01161291	0.01161292
100×10	0.00741472	0.00742623	0.00742623
convergence rate	$\mathcal{O}(h)$	$\mathcal{O}(h)$	$\mathcal{O}(h)$

Example 2. Consider the following semilinear parabolic equation:

$$\begin{cases} u_t - \Delta u + u^3 = g, & (X,t) \in \Omega \times J, \\ u = 0, & (X,t) \in \partial\Omega \times J, \\ u(0) = u_0, & (X,t) \in \Omega \times \{t = 0\}, \end{cases} \quad (14)$$

where $\Omega = (0,1) \times (0,1)$ and $J \in (0,1]$. g and u_0 are computed from the exact solution

$$u(x,y,t) = e^{-t+x+y} xy(1-x)(1-y).$$

Tables 3 and 4 show the numerical results at $t = 0.5$ and $t = 1.0$, respectively. The numerical results also support Theorems 3.1, 4.1, and 4.2. It is shown that the two-scale finite element discretizations, that is, Algorithms 4.1 and 4.2, are very efficient compared with the standard finite element discretization (6).

Table 3. *The errors at* $t = 0.5$. *Compare the two-scale finite element method with the standard finite element method.*

$\mathbf{1/h \times 1/H}$	$\|\mathbf{u^n - u^n_{h,h}}\|_1$	$\|\mathbf{u^n - (\tilde{u}^h_{H,H})^n}\|_1$	$\|\mathbf{u^n - (\tilde{u}^h_{H,H})^n}\|_1$
4×2	0.09193256	0.09878091	0.09877565
16×4	0.02326492	0.02393244	0.02394007
64×8	0.00582061	0.00586443	0.00589146
100×10	0.00372531	0.00374325	0.00378199
convergence rate	$\mathcal{O}(h)$	$\mathcal{O}(h)$	$\mathcal{O}(h)$

Table 4. *The errors at* $t = 1.0$. *Compare the two-scale finite element method with the standard finite element method.*

$\mathbf{1/h \times 1/H}$	$\|\mathbf{u^n - u^n_{h,h}}\|_1$	$\|\mathbf{u^n - (\tilde{u}^h_{H,H})^n}\|_1$	$\|\mathbf{u^n - (\tilde{u}^h_{H,H})^n}\|_1$
4×2	0.05576004	0.05991326	0.05991179
16×4	0.01411090	0.01451574	0.01451675
64×8	0.00353038	0.00355696	0.00356019
100×10	0.00225952	0.00227040	0.00227468
convergence rate	$\mathcal{O}(h)$	$\mathcal{O}(h)$	$\mathcal{O}(h)$

6. Conclusion

In this paper, the backward Euler two-scale finite element algorithms (Algorithms 4.1 and 4.2) for semilinear parabolic problems are proposed. Theoretical analysis and numerical examples show that on choosing $h = O(H^2)$ the backward Euler two-scale finite element approximations yield the same accuracy as the backward Euler standard finite element solution but much less computational cost. Besides, on the univariate fine grids, some semilinear problems are solved in Algorithm 4.1 while some linear problems are solved in Algorithm 4.2. Hence Algorithm 4.2 is even more efficient than Algorithm 4.1.

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